Particle Accelerators, 1995, Vol. 50, pp. 105–116
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LONGITUDINAL SINGLE BUNCH STABILITY

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(Received 23 January 1995; in final form 23 January 1995)

At high intensity, a short range wake field can distort the beam's potential well and thereby change the stationary distribution. It is now known that if this is not taken into account, instability thresholds will be incorrectly predicted. A numerical method exists for solving the linearized Vlasov equation for the self-consistent case, including the distortion to the stationary distribution, and finding such thresholds. We have found physical explanations for the eigenmodes and instability thresholds predicted in this method. As a result, a much simpler stability criterion has been found. The criterion is simple in that it depends only on the stationary distribution and does not require solution of the linearized Vlasov equation.

KEY WORDS: Instability, mode-coupling, bunch-lengthening

1 INTRODUCTION

The problem of longitudinal bunched beam stability is a complicated one; only limiting cases have been solved exactly. The narrow-band resonator can be treated because the long-range wake field does not appreciably distort the potential well. At the other extreme, i.e. space charge or inductive wall, the wake field is very short compared with the bunch length, and there exists a simple charge distribution (parabolic line density) for which there is no potential well distortion, though there is bunch-lengthening. As a result, there is no intensity-dependent synchrotron frequency spread and the problem has exact analytical solutions.[2] We have previously treated the case of space charge and arbitrary distributions, and have found that the criterion for stability is simply that the stationary distribution exist.[3]

Oide and Yokoya have suggested a method[4] which allows one to find eigenmodes and thresholds in the case of broad-band impedance. We call this the O+Y method. It includes the deformation of the equilibrium distribution due to the wake field. They found that the results obtained by this method are significantly different from those without including the deformation.

No rigorous analysis of the convergence and stability of the O+Y method has been made so far. Therefore, its limitations and precision are not known. However, the thresholds obtained seem to be in agreement with particle tracking simulations.

In this paper we derive an integral equation and show how the O+Y method can be derived from this equation. Also, analyzing just this integral equation we suggest a simple necessary criterion for single bunch stability and compare this method in the case of resonator impedance with results obtained by O+Y.[4]

The main feature of the new criterion is that the stability threshold in the case of only short-range forces can be obtained from analyzing the stationary distribution. This is similar to the case of space-charge impedance, where it was found that the stability threshold is equal to the threshold beyond which no stationary distribution can be found.[3]

An abbreviated version of the present paper was presented at the European Particle Accelerator Conference.[1]

2 THEORY

2.1 *Vlasov Equation*

The evolution of the distribution function $\psi(p, q)$ of charged particles in the bunch can be described by the Vlasov equation

$$\frac{\partial \psi}{\partial t} + p\frac{\partial \psi}{\partial q} + [-q + V(q, t)]\frac{\partial \psi}{\partial p} = 0, \tag{1}$$

where time has been normalized by the unperturbed synchrotron frequency, and the phase space variables (q, p) by their unperturbed rms values: $t \leftarrow \omega_{s0}t$, $p = (E - E_0)/\sigma_E$, and $q = z/\sigma_z$. $V(q)$ is the normalized induced force: it is written as a convolution of the wake function $W(q)$ with the line density $\lambda(q, t) = \int \psi dp$:

$$V(q, t) = -I\int_0^\infty W(q')\lambda(q + q', t)\, dq'. \tag{2}$$

The wake function can be found from the beam coupling impedance $Z(\omega)$:

$$W(q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikq} Z(\omega)\, d\omega, \tag{3}$$

where $q > 0$ and $k = \sigma_z\omega/c$. W has the dimension of impedance per time and the intensity parameter is given by $I = I_b/(\omega_{s0}\sigma_E/e)$. I_b is the current per bunch i.e. charge per bunch $\div$ revolution period.

The Vlasov equation (1) can be derived from $d\psi/dt = 0$ and the following Hamiltonian

$$H(q) = \frac{p^2}{2} + \frac{q^2}{2} + U_{\text{ind}}(q) \tag{4}$$

where the induced potential

$$U_{\text{ind}}(q) = -\int_0^\infty S(q')\lambda(q+q')\,dq' \tag{5}$$

is derived from the wake potential

$$S(q) = \int_0^q W(q')\,dq'. \tag{6}$$

The stationary distribution is $\psi = \psi_0(H)$ where ψ_0 is an arbitrary but well-behaved function. Phase space trajectories of individual particles are given by $H = \text{constant}$.

In the case of electrons, when synchrotron radiation becomes significant, an additional diffusion-like term should be added to the Vlasov equation (1). The resulting equation is the Fokker-Planck equation, and its time-independent solution is no longer an arbitrary function of H, but is the Maxwell-Boltzmann distribution $\psi_0(H) \propto e^{-H}$. Aside from its impact on the stationary distribution, the diffusion term is a small perturbation on the Vlasov equation in most cases, where the synchrotron frequency is large compared with the radiation damping time. The line density can be found from the Haïssinski equation using the method described by Zotter.[5] This gives the potential well by Eqn. 5, from which we find the phase space trajectories and incoherent synchrotron frequency as functions of H.

In the case of proton bunches, the stationary distribution is not necessarily Maxwell-Boltzmann if the acceleration or storage time is small compared with the relaxation time due to noise and radiation effects. In this case the Haïssinski equation does not apply. Nevertheless, the line density can be found by generalizing the above method, as for example we have previously[6] described for the case of an inductive wake.

2.2 *Eigenmode Integral Equation*

Let us now assume the distribution is perturbed from the stationary distribution: $\psi(p, q, t) = \psi_0(H(p, q)) + \psi_1(p, q, t)$. In the limit of small ψ_1, it is permissible to keep only terms linear in the perturbation and we can write the Vlasov equation in the form

$$\frac{\partial \psi_1}{\partial t} + p\frac{\partial \psi_1}{\partial q} + [-q + V_0(q)]\frac{\partial \psi_1}{\partial p} + V_1(q, t)\frac{\partial \psi_0}{\partial p} = 0 \tag{7}$$

It can be shown[4] that there is at least one solution of Eqn. 7:

$$\psi_1 = a\left(\frac{\partial \psi_0}{\partial q} - i\frac{\partial \psi_0}{\partial p}\right)e^{-it} \tag{8}$$

This is the so-called rigid dipole mode, and its frequency does not depend upon intensity.

Introducing canonical action $J = \frac{1}{2\pi}\oint p dq$ and angle θ ($H = H(J)$, $d\theta/dt = dH/dJ = \omega(J)$)[a] variables instead of p and q, we have

$$\frac{\partial \psi_1}{\partial t} + \omega(J)\frac{\partial \psi_1}{\partial \theta} + V_1(q,t)\frac{\partial \psi_0}{\partial p} = 0 \tag{9}$$

We will look for solutions of Eqn. 9 in the following form

$$\psi_1 = e^{-i\mu t}\sum_m C_m(J)\cos m\theta + S_m(J)\sin m\theta. \tag{10}$$

Substituting Eqn. 10 into Eqn. 9, one obtains the following integral equation:

$$\left[\mu^2 - m^2\omega^2(J)\right]C_m(J) = -m^2\omega^2(J)I\psi_0'(H(J))\sum_{m'}\int g_{m,m'}(J,J')C_{m'}(J')\,dJ' \tag{11}$$

where

$$g_{mm'}(J,J') = \frac{1}{\pi}\int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \,\cos m\theta\cos m'\theta'\, S(q(J,\theta) - q(J',\theta')). \tag{12}$$

The azimuthal mode index m extends from 1 to $\hat{m} = \infty$, but in our numerical calculations $\hat{m} = 3$ to 5 was usually sufficient.

At low intensities, when $J = H(p,q)$, the solution of the integral equation (11) for the rigid dipole mode $\mu = 1$ with the Maxwell-Boltzmann distribution is simply

$$\begin{aligned} C_1(H) &= \sqrt{H}\,\exp(-H) \\ C_m(H) &= 0, \quad m > 1 \end{aligned} \tag{13}$$

and this mode should be present among the solutions of Eqn. 11. This is a useful test of any numerical technique used to solve the integral equation.

2.3 Oide-Yokoya Method

A straightforward method for solving the integral Equation (11) is to convert it to a matrix equation: if the set $\{J_i\}$ is chosen so that $0 = J_0 < J_1 < \ldots < J_n$ then the matrix equation can be obtained from Equations 12 and 11 by making substitutions $C_m(J_n)\Delta J_n \rightarrow C_{mn}$. The resulting equation

$$\mu^2 C_{mn} = \sum_{m,n} M_{mnm'n'}C_{m'n'} \tag{14}$$

can then be solved numerically.

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[a] In order to keep the notation uncluttered, these frequencies have been normalized using the unperturbed synchrotron frequency ω_{s0}, and so are dimensionless. In cases like Eqn. 3, where ω refers to an rf frequency instead of a synchrotron frequency, it remains unnormalized.

Oide and Yokoya derived the above matrix equation by introducing a set of artificial orthogonal functions. Our analysis shows that their technique is equivalent to approximating an integral $\int f(x)dx$ by the sum $\sum f(x_i)\Delta x_i$. Such an approximation is only good for a sufficiently smooth integrand. Our results presented below show that the majority of the solutions found are not smooth. Therefore, their validity as collective modes is not obvious. A more rigorous mathematical analysis of the convergence and stability of the method is required. Nevertheless, we gain confidence from the fact that the instability thresholds obtained with this method are in good agreement with the particle tracking results presented by O+Y.[4]

Typically, to obtain stable results when the wake field is approximately equal to the bunch length requires at least 3 azimuthal (m) modes and 120 radial subdivisions (n), resulting in matrices at least 360 $\times$ 360 in size. For a given intensity, the stationary distribution has first to be found, next each matrix element has to be calculated by a double integration, and finally the matrix is analyzed for eigenvectors and eigenvalues. The CPU time required for the 360 $\times$ 360 case is around 2.5 minutes on a VAX 4000. The calculation is repeated for many intensities to find a possible threshold.

3 RESULTS

3.1 Capacitive Wake

Figure 1 (upper plot) shows eigenfrequencies μ vs. intensity I calculated in the case of a purely capacitive wake field $W(q) = \theta(q)/C$, where $\theta(q)$ is the Heaviside step function ($\theta(q) = 0$ for $q < 0$ and $\theta(q) = 1$ for $q \geq 0$), and C is the wake field capacitance. The Maxwell-Boltzmann distribution is known to be stable in this case at any intensity,[7] and indeed no complex eigenfrequencies are found.

One can see the rigid dipole mode clearly distinguished from the other eigenfrequencies because it happens that for this particular wake field all the other frequencies are shifted upward. We verified that the mode with $\mu = 1$ is indeed the rigid dipole mode by comparing $C_1(J)$ with that expected (13). See Figure 2.

What are the modes corresponding to the other eigenfrequencies in Figure 1? For each of the $\hat{n}$ values of J into which the problem has been subdivided, one can calculate the corresponding (incoherent) frequency $\omega(J_n)$. These and their integer multiples have been plotted in Figure 1 as well (lower plot). We see that they agree well with the other frequencies found by the O+Y method, excluding the rigid dipole mode. This indicates that these modes are not really collective modes. Further verification of this hypothesis comes from the fact that the eigenvector C^k_{mn} is found to be nonzero only at one or two values of n. This means that if the eigenvector has any physical interpretation at all, it represents an eigenfunction which is extremely localized in J, and becomes the narrower, the larger the matrix size. It should also be realized that the O+Y method forces the existence of $\hat{n}$ radial modes. We conclude, therefore, that these singular modes are not real, in the sense of being physically detectable. Moreover, we do not expect them to couple and thereby cause instability. We call these modes ‘incoherent’.

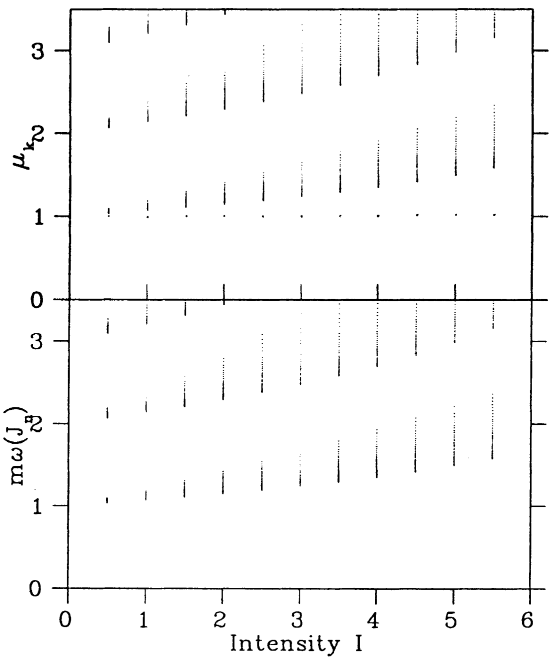

FIGURE 1: Frequencies vs. intensity I in the case of a purely capacitive impedance. The upper plot shows eigenfrequencies μ_k calculated by the O+Y method, and the lower shows incoherent frequencies $m\omega(J_n)$, where m is an integer. The eigenmode with frequency independent of intensity is the rigid dipole mode.

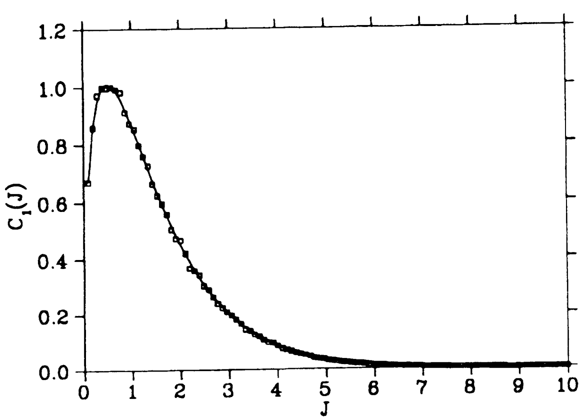

FIGURE 2: Comparison of the theoretical rigid dipole mode (13) (continuous curve) with the eigenvector C_{1n} of the μ=1 mode (□). The calculation is for $\hat{n}$=120, $\hat{m}$=3.

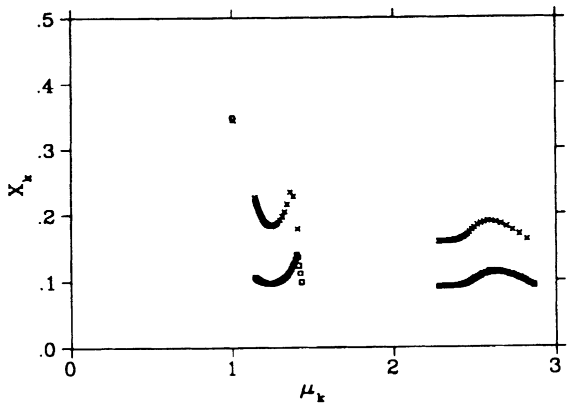

FIGURE 3: Rms value of the k^{th} eigenvector vs. its eigenvalue at intensity I=2 in the case of purely capacitive impedance with $\hat{n}$=40 (×) and $\hat{n}$=120 (□).

In order to separate real collective modes from the other incoherent eigenmodes, we define the normalized rms value of the k^{th} eigenvector C^k_{mn}:

$$X_k = \sqrt{\frac{1}{\hat{n}} \sum_{m,n} (C^k_{mn})^2}. \tag{15}$$

Since the eigenvectors C_{mn} are normalized to have a maximum value of 1, we then expect the narrow incoherent modes to have $X \ll 1$, while broad modes like the rigid dipole mode will have much larger X.

Figure 3 shows a plot of X_k versus μ_k for capacitive impedance. One can see that the point with maximum X also has $\mu = 1$, verifying that this is the rigid dipole mode. Also, this point is not sensitive to $\hat{n}$, whereas the other values of X all tend to zero as $\hat{n}$ is raised.

3.2 *Broad-band Wake*

Possible mode coupling which may lead to instability should take place between collective modes, not incoherent ones. One such mode is the rigid dipole mode, and the question is whether there are any others among the solutions of Eqn. 14 and whether they couple or not. We next try a resonator wake field given by impedance $Z(\omega) = R \Big/ \left[1 + iQ\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)\right]$ (where R is the shunt resistance and Q is the quality factor). Q should be small in order to approximate short range forces, and interesting effects are expected to occur when the resonant frequency ω_0 is comparable with the reciprocal of the bunch length, c/σ_z.

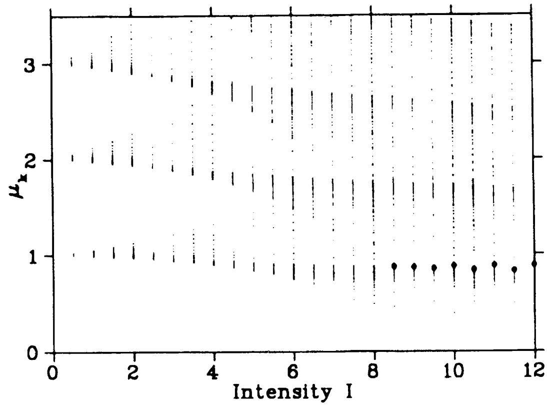

FIGURE 4: Eigenfrequencies vs. intensity for a broad-band type resonator (Q=1, k_0=0.6). Unstable modes are indicated by solid symbols.

The intensity I is defined as before, but now with the resonator's high frequency capacitance, $(\omega_0 R/Q)^{-1}$, used in place of $\mathcal{C}$. This is the same as the parameter S_r used by O+Y.[4] The parameter relating bunch length to wake length is $k_0 = \omega_0\sigma_z/c$. The calculated eigenfrequencies are shown in Figure 4. Note that there is an instability with a threshold of $I \approx 8$. As with the capacitive case, most of the frequencies correspond to incoherent modes. In fact, a plot of $m\omega(J_n)$ is virtually indistinguishable from Figure 4, except for the presence of the complex eigenfrequencies in Figure 4. The rigid dipole mode is not apparent because in the broad-band resonator case, incoherent frequencies are shifted both up and down, and so it is hidden among the incoherent modes. Close inspection of the eigenvectors reveals that those modes with frequencies near $\mu = 1$ all have more or less a rigid dipole component, depending upon how far they are from the frequency $\mu = 1$. Going to larger $\hat{n}$ is of no help, since there will always be a couple of incoherent modes near $\mu = 1$ which will heavily contaminate it.

If we plot the parameter X_k vs. μ_k (Figure 5) as before, the rigid dipole mode is still difficult to distinguish. However, there appear to be a few other 'real' modes as well. An investigation of the incoherent synchrotron frequency (Figure 6) shows that these modes are clustered near the local minimum. The physical interpretation is that near $d\omega/dJ = 0$ the particles can stay 'in step' longer, and so this area constitutes a 'coherent band' of action. The quadrupole mode is shown in Figure 7, where it is compared with an incoherent mode. Note the dramatic difference: the coherent mode is smooth and independent of $\hat{n}$, whereas the incoherent mode is narrow and very sensitive to $\hat{n}$. The reason that the coherent mode appears so unambiguously is of course that it occurs at the minimum of the synchrotron frequency and so is not degenerate with any incoherent modes.

As intensity increases, $\omega_{\min}$ decreases (Figure 6), and just at threshold it is near $1/2$. This suggests that the instability arises because of coupling of the quadrupole mode located in the 'coherent band' with the rigid dipole mode. This conjecture is verified by an inspection of the eigenvector of the unstable mode. Also, by extrapolating the lowest frequency quadrupole mode in Figure 4, we see it crosses the rigid dipole mode near the threshold intensity.

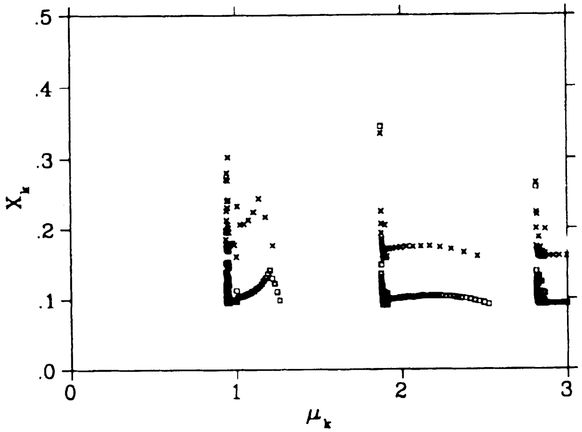

FIGURE 5: Rms value of the k^{th} eigenvector vs. its eigenvalue at intensity $I=3$ in the case of the broad-band resonator with $Q=1$, $k_0=0.6$ for two different matrix sizes: $\hat{n}=40$ ($\times$) and $\hat{n}=120$ ($\Box$). Note that there are two modes whose X is independent of $\hat{n}$; these correspond to resp. a quadrupole and a sextupole mode at the location where $\omega(J)$ is a minimum.

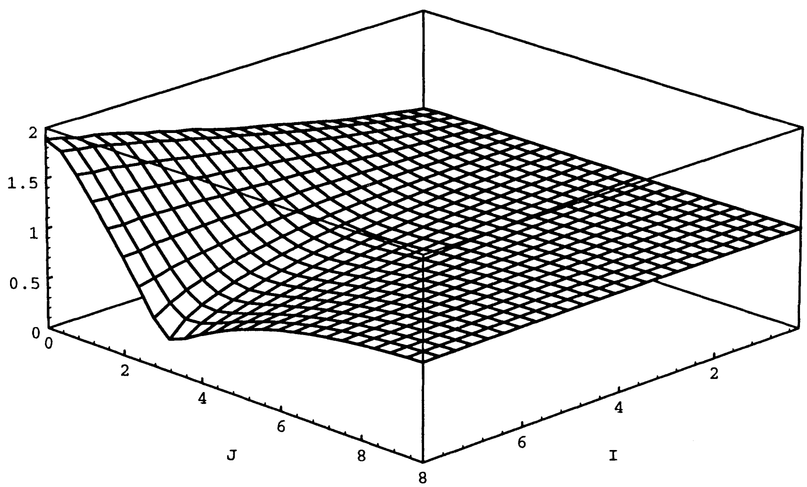

FIGURE 6: Normalized synchrotron frequency vs. action J and intensity I for the resonator impedance $Q=1$, $k_0=0.6$.

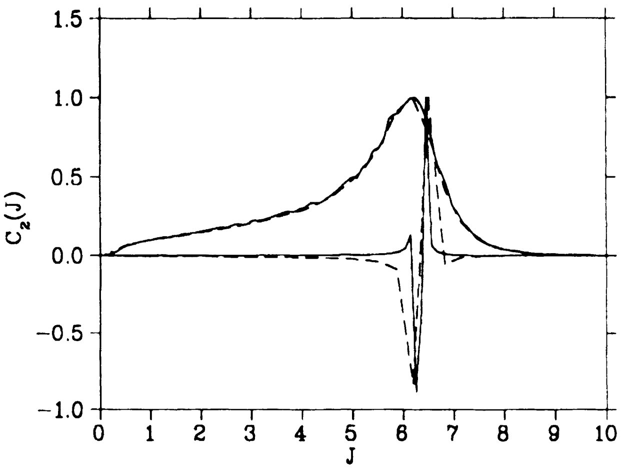

FIGURE 7: Comparison of the 'coherent' quadrupole mode at the synchrotron frequency minimum (μ=$2\omega_{\min}$) with a nearby 'incoherent' quadrupole mode, for two different matrix sizes: $\hat{n}$=120 (continuous curve) and $\hat{n}$=40 (dashed curve).

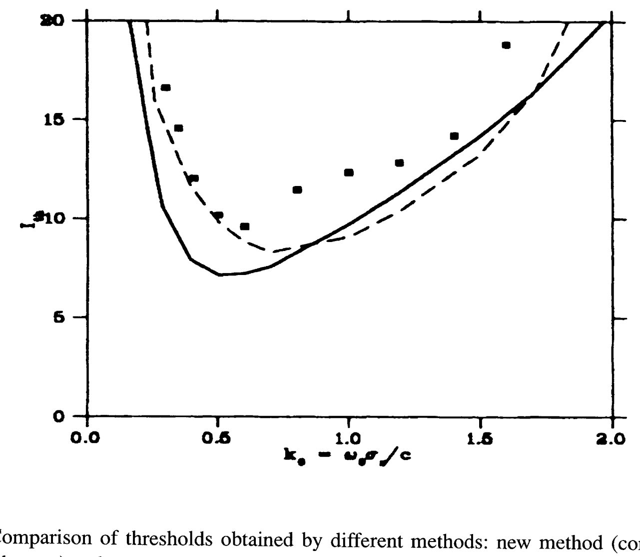

FIGURE 8: Comparison of thresholds obtained by different methods: new method (continuous curve), O+Y method (dashed curve) and numerical tracking (symbols). The latter two cases are taken from O+Y.[4].

The threshold calculated from the criterion $\omega_{\min} = 1/2$ has been plotted vs. the bunch length in Figure 8 on top of the data from O+Y.[4] The agreement is good: the discrepancy between the solid and dashed curves is probably due to the truncation of the matrix in the latter case.

4 CONCLUSIONS

4.1 Oide-Yokoya Method

Oide and Yokoya have invented a helpful technique for solving a previously intractable problem. However, the method should be used with caution. It always generates as many eigenmodes as the rank of the matrix. The vast majority of these modes appear to be infinitesimally narrow in the limit of infinite rank, and have eigenfrequencies equal to the incoherent synchrotron frequency at the mode location. The method does find real collective modes such as the rigid dipole mode, but these are difficult to extract in cases where they are degenerate with the incoherent modes. Nevertheless, even in degenerate cases instability thresholds attributable to coupling between real collective modes are found, and these appear to be valid.

4.2 Landau damping

The fact that so few real modes are found should not be surprising. Landau-damped excitations do not have exponential time dependence, so the standard method of solving the Vlasov equation by assuming harmonic time dependence simply should not find any modes which we think of as being 'Landau-damped'. The situation is similar to the case where synchrotron frequency spread originates externally by rf wave nonlinearity. Besnier[8] originally developed a technique for solving the Sacherer integral equation where the dispersion function was expanded in the same orthogonal polynomials as the kernel. The technique has the same problem as the O+Y method in that there will always be as many modes as the size of the resulting matrix. Subsequent reanalysis by Chin, Satoh, and Yokoya[9] showed that below threshold these modes had growth rates which tended to zero as matrix size was increased. They developed a dispersion integral approach and found that below threshold there simply were no modes, while above threshold their eigenfrequencies agreed with those found in the Besnier technique.

The difference between external frequency spread and that arising from the wake field itself is that in the latter case the frequency 'spread' and the mode 'shift' change at the same rate with intensity. In the former case, the modes which would exist in the absence of frequency spread are one-by-one 'freed' from the incoherent band as the intensity is raised. But in the latter case, the dispersion also grows with intensity and so most of the modes which would exist in the absence of dispersion do not exist (equivalently, they are Landau-damped) at any intensity. The few modes that do exist can be unstable on their own (for example, with a resistive wake, a case not investigated here), or can become unstable by coupling together when their frequencies coincide.

4.3 *New Criterion*

A simple necessary criterion for stability suggested in this paper is in reasonable agreement with the O+Y method and numerical tracking. The instability threshold can be found by analyzing the stationary self-consistent distribution, without solving the Vlasov equation. One collective mode found in this method is the rigid dipole mode and the others are multipole 'collective' modes concentrated near the synchrotron amplitude where the synchrotron frequency is a minimum. Since this minimum $\omega_{\rm min}$ decreases as intensity is raised, and the frequency of the rigid dipole mode is a constant ($= 1$), it may happen that $m\omega_{\rm min} = 1$, at which point coupling between the rigid dipole and the m multipole mode leads to instability. The intensity at which $\omega_{\rm min} = 1/2$, thus corresponds to an instability threshold. This is similar to, but more stringent than, the threshold suggested by P. Wilson as quoted by Bane and Oide:[10] namely that instability occurs when $\omega_{\rm min} = 0$.

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